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Higher Weights of Anticodes and the Generalized Griesmer Bound*

Dmitrii Yu. Nogin

Institute for Information Transmission Problems, Moscow, Russia
E-mail: nogin@iitp.ru

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We show that the minimum r-weight d_r of an anticode can be expressed in terms of the maximum r-weight of the corresponding code. As examples, we consider anticodes from homogeneous hypersurfaces (quadrics and Hermitian varieties). In a number of cases, all differences (except for one) of the weight hierarchy of such an anticode meet an analog of the generalized Griesmer bound. © 1999 Academic Press

1. INTRODUCTION

A standard way of constructing a q-ary linear code from an algebraic variety is to regard the set of $\mathbb{F}_q$-rational points of the variety as a projective system that defines a code (see, e.g., [1–3]). Recall [4, Sect. 1.1.2] (for details, see [5]) that a projective $[n, k]$ system is a set of n points in a projective space $\mathbb{P}^{k-1}$ of dimension $(k-1)$ over $\mathbb{F}_q$. There exists a natural one-to-one correspondence between equivalence classes of nondegenerate $[n, k]$ systems and equivalence classes of nondegenerate linear $[n, k]$ codes. Under this correspondence, points of a system (more precisely, arbitrary k-vectors whose projectivizations are the points of the system) correspond to columns of a generator matrix of a code, codewords correspond to hyperplanes in $\mathbb{P}^{k-1}$, and the weight of a codeword (a subcode of dimension r) is the number of points of the system that lie outside the corresponding hyperplane (plane of codimension r).

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However, if the number of $\mathbb{F}_q$-rational points of a variety is large enough (which is the case, for instance, if the variety is a homogeneous hypersurface—then the system contains, roughly speaking, $1/q$ times the number of all points of the space), it will be a good plan to take as a projective system not the points of the variety but, on the contrary, the remaining points of the space. In fact, this is a standard construction of an anticode [6, Sect. 17.6], which can be described in terms of generator matrices as follows: Consider the matrix which consists of all different (up to multiplying by a constant) nonzero columns of length k, that is, the parity-check matrix of a Hamming code, and then delete all columns that are columns of the generator matrix of C. The remaining columns form the generator matrix of the anticode $\bar{C}$. It should be noted that in the present paper we consider only projective systems without multiple points, that is, codes without repetitions.

In Section 2, we show how to compute generalized weights of an anticode. In Section 3, we consider examples of anticodes from quadrics and Hermitian varieties. In a number of cases, weight hierarchies of these codes have certain interesting properties; namely, the equality

$$d_{i+1} - d_i = \left\lceil \frac{d_i - d_{i-1}}{q} \right\rceil$$

holds for all i except for one. In more detail, these properties are considered in Section 4.

2. PARAMETERS OF ANTICODES

Let C be an $[n, k]_q$ code, let X be the corresponding projective system of n points in $\mathbb{P}^{k-1} = \mathbb{P}(\mathbb{F}_q)^k$, and let $d_r = d_r(C)$ be higher weights, $r = 1, \ldots, k$. Recall [5] that in the language of projective systems

$$d_r = d_r(X) = \min_{H^r} |X \backslash H^r| = n - \max_{H^r} |X \cap H^r|, \tag{1}$$

where $H^1 = H$ is an arbitrary hyperplane and H^r is an arbitrary subspace of codimension r in $\mathbb{P}^{k-1}$.

Consider the projective system $\bar{X} = \mathbb{P}^{k-1} \backslash X$ and the corresponding anticode $\bar{C}$. It is clear that the length $n(\bar{C})$ of the anticode equals $|\mathbb{P}^{k-1} \backslash X| = \theta_{k-1} - n$, where $\theta_a = (q^{a+1} - 1)/(q-1)$ is the number of points of the projective space $\mathbb{P}^a$. It is also clear that if $\bar{X}$ does not entirely lie in a hyperplane (which always is the case, e.g., if $n < q^{k-1}$), then $\dim \bar{C} = k$.

Let us consider $d_r(\bar{C}) = d_r(\bar{X})$. According to (1),

$$d_r(\bar{C}) = n(\bar{C}) - \max_{H^r} |(\mathbb{P}^{k-1} \backslash X) \cap H^r|.$$

Here, $n(\bar{C}) = \theta_{k-1} - n$ and

$$(\mathbb{P}^{k-1} \backslash X) \cap H^r = (\mathbb{P}^{k-1} \cap H^r) \backslash (X \cap H^r) = H^r \backslash (X \cap H^r);$$

that is,

$$\begin{aligned} d_r(\bar{C}) &= \theta_{k-1} - n - \max_{H^r} (|H^r| - |X \cap H^r|) \\ &= \theta_{k-1} - |H^r| - n + \min_{H^r} |X \cap H^r| \\ &= \theta_{k-1} - \theta_{k-r-1} - \left(n - \min_{H^r} |X \cap H^r|\right) \\ &= q^{k-r}\theta_{r-1} - \max_{H^r} |X \backslash H^r|, \end{aligned}$$

where $\max_{H^r} |X \backslash H^r| = d_{r,\max}(C)$ is the maximum r-weight of C (cf. (1)). Here we used the obvious equality $\theta_a - \theta_b = q^{b+1}\theta_{a-b-1}$. Thus, we have obtained the following statement.

PROPOSITION 1. *Let C be a nondegenerate $[n, k]_q$ code without repetitions. Assume that the corresponding projective system is such that the complement of it does not lie in a hyperplane. Then the anticode $\bar{C}$ is a $[\theta_{k-1} - n, k]_q$ code with higher weights*

$$d_r(\bar{C}) = q^{k-r}\theta_{r-1} - d_{r,\max}(C). \tag{2}$$

Remark 1. Note that in the case where the complete higher weight distribution (generalized weight spectrum) is known, with the help of similar reasoning one can easily compute the distribution of higher weights of the corresponding anticode.

3. EXAMPLES

Relation (2) makes it possible to find the weight hierarchy of an anticode in the case where the maximum r-weights of a code are known. As examples, we consider codes associated with homogeneous hypersurfaces, namely, quadrics [2] and Hermitian varieties [1], since for these codes the higher weight distribution is known.

3.1. *Anticodes from Quadrics*

Weight hierarchies and generalized spectra of codes from quadrics are described in [2] (see also [7]). Let us recall in brief that a nonsingular quadric $\mathcal{Q}_{k-1}$ in $\mathbb{P}^{k-1}$ is the set of zeros of a nonsingular quadratic form in k variables (see, e.g., [8, Chap. 22]). Quadrics can be of three types modulo the projective equivalence, namely, hyperbolic ($\mathcal{H}_{k-1}$), parabolic ($\mathcal{P}_{k-1}$), and elliptic quadrics ($\mathcal{E}_{k-1}$). Hyperbolic and elliptic quadrics exist for even k, and parabolic for odd k. The character of a nonsingular quadric is the number $w = 2g - k + 4$, where g is the (projective) index of the quadric, that is, the projective dimension of the largest linear space lying in the quadric. Thus, quadrics of the hyperbolic type have character $w = 2$, of the parabolic type, character $w = 1$, and of the elliptic type, character $w = 0$.

It is shown in [2] that the lengths of the corresponding codes of dimension k are,

$$\begin{array}{lll} \text{for a hyperbolic quadric,} & n = \theta_{k-2} + q^{k/2-1} & (k \text{ is even}), \\ \text{for a parabolic quadric,} & n = \theta_{k-2} & (k \text{ is odd}), \\ \text{for an elliptic quadric,} & n = \theta_{k-2} - q^{k/2-1} & (k \text{ is even}), \end{array}$$

that is, $n = \theta_{k-2} + (w-1)q^{k/2-1}$. It is also shown [2] that all r-weights of these codes are the numbers

$$q^{k-r-1}\theta_{r-1} + (w-1)q^{k/2-1} - q^{k-r-2}(v-1)\,q^{(1-t)/2}, \tag{3}$$

where t and v are integers such that

$$\begin{gathered} k - r - 1 \geq t \geq \max\{k - 2r - 1 + |w - v|, 1 - v\}, \\ 2 \geq v \geq 0, \quad t \text{ and } v \text{ are of different parity.} \end{gathered} \tag{4}$$

The geometric meaning of the numbers t and v is as follows: a section of a nonsingular quadric $\mathcal{Q}_{k-1}$ by a subspace H^r is a cone $H^{r+t+1}\mathcal{Q}_t$ with the vertex H^{r+t+1} over a nonsingular quadric $\mathcal{Q}_t$ of character v; conditions (4) are the conditions for such sections to exist.

THEOREM 1. *For $k \geq 3$, an anticode $\bar{C}$ associated with a nonsingular quadric $\mathcal{H}_{k-1}$ of character w is a code of dimension k with the following parameters*:

in the hyperbolic case ($w = 2$, *k is even*),

$$n = q^{k-1} - q^{k/2-1}, \quad d_r = \begin{cases} q^{k-1} - q^{k-2} - q^{k/2-2}, & r = 1, \\ q^{k-1} - q^{k-r-1} - q^{k/2-1} - q^{k/2-2}, & 2 \le r \le k/2, \\ q^{k-1} - q^{k/2-1} - q^{k-r-1} - q^{k-r-2}, & k/2 \le r \le k-2, \\ q^{k-1} - q^{k/2-1} - 1, & r = k-1, \\ q^{k-1} - q^{k/2-1}, & r = k; \end{cases} \tag{5}$$

in the parabolic case ($w = 1$, *k is odd*),

$$n = q^{k-1}, \quad d_r = \begin{cases} q^{k-1} - q^{k-r-1} - q^{(k-1)/2-1}, & 1 \le r \le (k-1)/2, \\ q^{k-1} - q^{k-r-1} - q^{k-r-2}, & (k-1)/2 \le r \le k-2, \\ q^{k-1} - 1, & r = k-1, \\ q^{k-1}, & r = k; \end{cases} \tag{6}$$

in the elliptic case ($w = 0$, *k is even*),

$$n = q^{k-1} - q^{k/2-1},$$

$$d_r = \begin{cases} q^{k-1} - q^{k-r-1}, & 1 \le r \le k/2-1, \\ q^{k-1} + q^{k/2-1} - q^{k-r-1} - q^{k-r-2}, & k/2-1 \le r \le k-2, \\ q^{k-1} + q^{k/2-1} - 1, & r = k-1, \\ q^{k-1} + q^{k/2-1}, & r = k. \end{cases} \tag{7}$$

Proof. Let us apply Proposition 1. To compute $d_{r,\max}$, we have to maximize (3) over all pairs (t, v) that satisfy (4). Thus,

$$d_{r,\max}(C) = q^{k-r-1}\theta_{r-1} + (w-1)q^{k/2-1} - q^{k-r-2} \min_{(t,v)} \frac{v-1}{q^{(t-1)/2}},$$

whence, using (2), we obtain

$$d_r(\bar{C}) = q^{k-1} - q^{k-r-1} - (w-1)q^{k/2-1} + q^{k-r-2} \min_{(t,v)} \frac{v-1}{q^{(t-1)/2}}. \tag{8}$$

The minimum in (8) is attained at $v = 0$ if this is possible; here, t should be as small as possible. If $v \neq 0$, the minimum is attained at $v = 1$ (if this is possible), in this case t is arbitrary. Finally, if $v=2$, t should be as large as possible. Let us consider three cases for different types of quadrics separately.

I. Hyperbolic case: $w=2$, k is even. Under $v=0$, conditions (4) take the form

$$k-r-1 \geq t \geq k-2r+1,$$
$$t \geq 1, \quad t \text{ is odd},$$

whence

$$\text{for } 2 \leq r \leq k/2, \quad t=k-2r+1,$$
$$\text{for } k/2 \leq r \leq k-2, \quad t=1;$$

and if $r=1$, we have to take $v=1$ and then the conditions (4) have the form

$$t=k-2, \quad t \geq 0, \quad t \text{ is even},$$

whence $t=k-2$. Similarly, for $r=k-1$, we get $v=1$, thus $t=0$; for $r=k$, we get $v=2$, $t=-1$.

Taking account of this and using (8), we obtain (5).

II. Parabolic case: $w=1$, k is odd. Under $v=0$, conditions (4) take the form

$$k-r-1 \geq t \geq k-2r,$$
$$t \geq 1, \ t \text{ is odd},$$

whence

$$\text{for } 2 \leq r \leq \frac{k-1}{2}, \qquad t=k-2r,$$

$$\text{for } \frac{k-1}{2} \leq r \leq k-2, \quad t=1;$$

and if $r=1$, we take $v=1$, then $t=0$; for $r=k$, we get $v=2$, $t=-1$. Substituting these pairs (t,v) into (8), we obtain (6).

III. Elliptic case: $w=0$, k is even. Under $v=0$, conditions (4) take the form

$$k-r-1 \geq t \geq k-2r-1,$$
$$t \geq 1, \quad t \text{ is odd},$$

whence

$$\begin{aligned} &\text{for } 2 \le r \le k/2-1, && t = k-2r-1,\\ &\text{for } k/2-1 \le r \le k-2, && t = 1; \end{aligned}$$

for $r = k-1$, we get $v = 1$, $t = 0$; for $r = k$, we get $v = 2$, $t = -1$. Substituting this into (8), we obtain (7). ∎

Recall that the Griesmer bound for generalized weights is $d_r \ge \sum_{i=0}^{r-1} \lceil d_1/q^i \rceil$. In [2], it is shown that higher weights of codes from hyperbolic quadrics meet this bound for $r \le k/2$. In more detail, we discuss this property in Section 4, and now we note that a similar property holds for anticodes from elliptic quadrics as well.

COROLLARY 1. *For $r \le k/2-1$, the weights d_r of an anticode from an elliptic quadric meet the Griesmer bound.*

Proof. Indeed, from (7) we get $d_1 = q^{k-1} - q^{k-2}$, which immediately implies the desired equality. ∎

3.2. *Anticodes from Hermitian Varieties*

Weight hierarchies and generalized spectra of Hermitian codes are computed in [1]. Recall that a (nonsingular) Hermitian variety [8, Chap. 23] is the set of zeros of a (nonsingular) Hermitian form $\sum X_i \bar{X}_i$, where $\bar{X} = X^{\sqrt{q}}$ (here, q is a square).

The properties of Hermitian varieties $\mathcal{U}_{k-1}$ in $\mathbb{P}^{k-1}$ are slightly different for even and odd k. In [1], it is shown that the corresponding codes of dimension k have lengths

$$\begin{aligned} n &= \theta_{k/2-1}\,(q^{(k-1)/2}+1),\ k \text{ even},\\ n &= \theta_{(k-1)/2-1}\,(q^{k/2}+1),\quad k \text{ odd}. \end{aligned}$$

It is also shown there that all possible r-weights of these codes are of the form

$$n - \theta_{k-r-2} - \frac{(q^{k-r-1} - (-\sqrt{q})^{k+v-r})}{\sqrt{q}+1}, \tag{9}$$

where v is an integer such that

$$-1 \le v \le \min\{r-1,\ k-r-1\}. \tag{10}$$

The geometric meaning of the number v is as follows: a section of a nonsingular Hermitian variety $\mathcal{U}_{k-1}$ by a subspace H^r is a cone $H^{k-1-v}\mathcal{U}_{k-2-r-v}$ over a nonsingular Hermitian variety $\mathcal{U}_{k-2-r-v}$, that is, v equals the codimension of the vertex of this cone; condition (10) is the condition for such sections to exist.

THEOREM 2. *An anticode $\bar{C}$ associated with a nonsingular Hermitian variety $\mathcal{U}_{k-1}$ is a code of dimension k with the following parameters*:
for even k,

$$n = q^{k/2}\theta_{k/2-1}\left(1 - \frac{1}{\sqrt{q}}\right),$$

$$d_r = \begin{cases} q^{k-r}\theta_{r-1}\left(1 - \frac{1}{\sqrt{q}}\right) - q^{k/2-1}, & 1 \le r \le k/2, \\ n - q^{k-r-1}, & k/2 \le r \le k-1, \\ n, & r = k; \end{cases} \tag{11}$$

for odd k,

$$n = q^{(k-1)/2}\left(\theta_{(k-1)/2} - \theta_{(k-1)/2-1}\sqrt{q}\right),$$

$$d_r = \begin{cases} q^{k-r}\theta_{r-1}\left(1 - \frac{1}{\sqrt{q}}\right), & 1 \le r \le \frac{k-1}{2}, \\ n - q^{k-r-1}, & \frac{k-1}{2} \le r \le k-1, \\ n, & r = k; \end{cases} \tag{12}$$

The proof is similar to that of Theorem 1. According to Proposition 1, we have to compute $d_{r,\max}$. To do this, we should maximize (9) over all v that satisfy (10). Thus, by (2) and (9), we obtain

$$d_r(\bar{C}) = q^{k-r}\theta_{r-1} - n(C) + \theta_{k-r-2} + \frac{q^{k-r-1} - \max\limits_{v}(-\sqrt{q})^{k+v-r}}{\sqrt{q}+1}. \tag{13}$$

The maximum in (13) is attained if $k + v - r$ is even (if there exist such v's satisfying (10)); here, v should be as large as possible. Let us consider the cases of odd and even k separately.

Assume that k is even. To maximize (13), we should take v of the same parity as r whenever possible. Then condition (10) implies that the maximum in (13) is attained at

$$v = r - 2, \quad 1 \le r \le k/2,$$

$$v = k - r - 2, \quad k/2 \le r \le k - 1,$$

$$v = -1, \quad r = k.$$

Substituting these values of v into (13), after some transformations we obtain (11).

Now, assume that k is odd. To maximize (13), we should take v of parity different from that of r whenever possible. Then (10) implies that the maximum is attained at

$$v = r - 1, \quad 1 \le r \le \frac{k-1}{2},$$

$$v = k - r - 2, \quad \frac{k-1}{2} \le r \le k - 1,$$

$$v = -1, \quad r = k.$$

Substituting these v into (13), after some transformations we obtain (12).

Theorem 2 yields, in particular, the following observation. In [1], it is shown that for even k, the weights d_r of a Hermitian code meet the Griesmer bound if $r < k/2$. For odd k, this is not the case; however, for anticodes, the similar property holds.

COROLLARY 2. *For $r \le (k-1)/2$, the weights d_r of an anticode of odd dimension k from a Hermitian variety meet the Griesmer bound.*

Indeed, we obtain from (12) that $d_1 = q^{k-1}(1 - 1/\sqrt{q})$, which implies the desired statement.

4. DIFFERENCES OF WEIGHT HIERARCHIES AND THE GRIESMER BOUND

Thus, there are examples of codes whose higher weights d_r meet the Griesmer bound for r up to (about) half the code dimension. These are codes from hyperbolic quadrics, anticodes from elliptic quadrics, Hermitian codes for even k, and Hermitian anticodes for odd k. We want to rationalize the structure of the remaining parts of weight hierarchies in these cases.

For the convenience of the description of weight hierarchies, we consider not the weights d_r themselves, but the hierarchy differences, that is, the quantities $\Delta_r = d_r - d_{r-1}$ (it is quite natural to assume $d_0 = 0$, which conforms with the definition (1) well; that is, $\Delta_1 = d_1$). As will be shown below, these quantities obey remarkable properties in a number of cases.

First, recall that a code satisfies the so-called Chain Condition if all of its minimum generalized weights are attained at a sequence of embedded subspaces (subcodes). One can easily check that all codes mentioned in this paper (codes and anticodes from quadrics and Hermitian varieties) satisfy the Chain Condition. To verify this, it suffices to note that all sections of the corresponding varieties are cones and the dimensions of vertices of these cones increase in r—see formulas (4) and (10) and remarks to them. The values of the parameters t and v are found in the proofs of Theorems 1 and 2 and the corresponding theorems in [1, 2] (a detailed check for the case of quadrics is performed in [7]).

Thus, in particular, for a hyperbolic quadric $\mathscr{H}_{k-1}$, the sequence of the embedded minimum-weight sections has the form

$$\begin{aligned} \mathscr{H}_{k-1} \supset \Pi_0 \mathscr{H}_{k-3} \supset \Pi_1 \mathscr{H}_{k-5} \supset \cdots \supset \Pi_{k/2-1} \mathscr{H}_{-1} \\ = \Pi_{k/2-1} \supset \Pi_{k/2-2} \supset \cdots \supset \Pi_0 \end{aligned} \tag{14}$$

(here and in what follows, we use the notation Π_v for a plane of dimension v, that is, $\Pi_v = H^{k-1-v}$ and Π_0 is a point); for a parabolic quadric,

$$\mathscr{P}_{k-1} \supset \mathscr{H}_{k-2} \supset \Pi_0 \mathscr{H}_{k-4} \supset \cdots; \tag{15}$$

for an elliptic quadric,

$$\mathscr{E}_{k-1} \supset \mathscr{P}_{k-2} \supset \mathscr{H}_{k-3} \supset \Pi_0 \mathscr{H}_{k-5} \supset \cdots; \tag{16}$$

for a Hermitian variety $\mathscr{U}_{k-1}$, for even k,

$$\begin{aligned} \mathscr{U}_{k-1} \supset \Pi_0 \mathscr{U}_{k-3} \supset \Pi_1 \mathscr{U}_{k-5} \supset \cdots \supset \Pi_{k/2-1} \mathscr{U}_{-1} \\ = \Pi_{k/2-1} \supset \Pi_{k/2-2} \supset \cdots \supset \Pi_0; \end{aligned} \tag{17}$$

for odd k (thereby, for even $k-1$),

$$\mathscr{U}_{k-1} \supset \mathscr{U}_{k-2} \supset \Pi_0 \mathscr{U}_{k-4} \supset \cdots . \tag{18}$$

Sequences of embedded maximum-weight sections (that is, sections corresponding to minimum-weight codewords in anticodes) for these varieties are as follows: for an elliptic quadric,

$$\begin{aligned} \mathscr{E}_{k-1} \supset \Pi_0 \mathscr{E}_{k-3} \supset \Pi_1 \mathscr{E}_{k-5} \supset \cdots \supset \Pi_{k/2-2} \mathscr{E}_1 \\ = \Pi_{k/2-2} \supset \Pi_{k/2-3} \supset \cdots \supset \Pi_0 \supset \varnothing \supset \varnothing; \end{aligned} \tag{19}$$

for a parabolic quadric,

$$\mathscr{P}_{k-1} \supset \mathscr{E}_{k-2} \supset \Pi_0 \mathscr{E}_{k-4} \supset \cdots; \tag{20}$$

for a hyperbolic quadric,

$$\mathscr{H}_{k-1} \supset \mathscr{P}_{k-2} \supset \mathscr{E}_{k-3} \supset \Pi_0 \mathscr{E}_{k-5} \cdots; \tag{21}$$

for a Hermitian variety, for odd k,

$$\begin{aligned} &\mathscr{U}_{k-1} \supset \Pi_0 \mathscr{U}_{k-3} \supset \Pi_1 \mathscr{U}_{k-5} \supset \cdots \supset \Pi_{(k-1)/2-1} \mathscr{U}_0 \\ &= \Pi_{(k-1)/2-1} \supset \Pi_{(k-1)/2-2} \supset \cdots \supset \Pi_0 \supset \varnothing; \end{aligned} \tag{22}$$

for even k (thereby, odd $k-1$),

$$\mathscr{U}_{k-1} \supset \mathscr{U}_{k-2} \supset \Pi_0 \mathscr{U}_{k-4} \supset \cdots . \tag{23}$$

Here, the chain (15) starting from the second term and the chain (16) starting from the third term have precisely the same structure as (14); the chain (18) starting from the second term, as (17); (20) from the second term and (21) from the third term, as (19); and (23) from the second term, as (22). The argument below explains the relation between sequences of differences of weight hierarchies for chains which differ by several starting terms. Due to this relation, in what follows it suffices to consider only the chains (14), (17), (19), and (22).

Consider an arbitrary code C satisfying the Chain Condition. Consider also the corresponding projective system X. Let H be a hyperplane at which the minimum weight is attained. Consider the system $X' = X \cap H$ as a projective system in $\mathbb{P}^{k-2} = H$. Then, as is readily seen, $d_r(X') = d_{r+1}(X) - d_1(X)$; that is, for the hierarchy differences we have $\Delta_r(X') = \Delta_{r+1}(X)$. Thus, the sequence of differences of the weight hierarchy for the new system is the same, but with the first term, $\Delta_1(X) = d_1(X)$, deleted. In the code language, C' is obtained from C by puncturing all nonzero coordinates of a minimum-weight codeword.

We can repeat this procedure several times. Applying the Griesmer bound $d_2 \geq d_1 + \lceil d_1/q \rceil$ to systems thus obtained, we arrive at

$$\Delta_{r+1} \geq \left\lceil \frac{\Delta_r}{q} \right\rceil. \tag{24}$$

It is natural to call the latter inequality the *difference Griesmer bound*; successively summing up inequalities (24) starting from $r = 1$, we obtain the ordinary Griesmer bound for generalized weights. For the binary case and in different notation, this inequality was first presented in [9]. For the q-ary case and in slightly different notations, these inequalities are given, e.g., in [10]. As is shown below, for codes that do not satisfy the Chain Condition, inequalities (24) need not hold.

The relation between the difference Griesmer bound, the Griesmer bound for generalized weights, and the Chain Condition is described in the proposition below.

PROPOSITION 2. *Consider the following properties of codes*:

(a) *the Chain Condition*;

(b) *the difference Griesmer bounds holds for all* r, *that is*, $\Delta_{r+1} \geq \lceil \Delta_r/q \rceil$ *for all* r;

(c) *the differences of the weight hierarchy meet the difference Griesmer bound for all* r, *that is*, $\Delta_{r+1} = \lceil \Delta_r/q \rceil$ *for all* r;

(d) *all weights of the code meet the generalized Griesmer bound*, *that is*, $d_r = \sum_{i=0}^{r-1} \lceil d_1/q^i \rceil$ *for all* r;

(e) *the code meets the ordinary Griesmer bound*, *that is*, $n = \sum_{i=0}^{k-1} \lceil d_1/q^i \rceil$.

The following diagram shows their connections:

$$\begin{array}{ccccc} & (a) & \rightarrow & (b) & \\ & \nwarrow & & \nearrow & \\ (c) & \Leftrightarrow & (d) & \Leftrightarrow & (e) \end{array}$$

Arrows denote implications (*single arrow means that the converse implication is not valid*). *Condition* (b) *is not tautological* (*nor all codes satisfy* (b)).

Proof. Let us begin with the bottom part of the diagram. The equivalence (c) $\Leftrightarrow$ (d) is obvious since both conditions mean that $\Delta_r = \lceil d_1/q^{r-1} \rceil$. The implication (d) $\Rightarrow$ (e) is trivial since $n = d_k$.

Implications (e) $\Rightarrow$ (d) and (e) $\Rightarrow$ (a) are proved in [11] and [12, 13] respectively; we present a proof of both implications simultaneously in geometric language. More precisely, let us show that if H^r is a subspace of the minimum weight $d_r^{\mathrm{Gr}} = \sum_{i=0}^{r-1} \lceil d_1/q^i \rceil$ which meets the Griesmer bound, then among the subspaces H^{r-1} passing through H^r there is at least one of weight d_{r-1}^{Gr}. Assume the contrary—let each of them have weight wt $H^{r-1} \geq d_{r-1}^{\mathrm{Gr}} + 1$. Each of the subspaces H^{r-1} contains $d_r - \mathrm{wt}\, H^{r-1}$ points of the projective system that lie outside H^r, $\bigcup (H^{r-1} \setminus H^r) = \mathbb{P}^{k-1} \setminus H^r$; therefore, $\sum (d_r - \mathrm{wt}\, H^{r-1}) = d_r$, where the join and sum are taken over all H^{r-1} passing through H^r. The number of such H^{r-1} equals θ_{r-1}. By the assumption,

$$d_r - \mathrm{wt}\ H^{r-1} \leq d_r^{\mathrm{Gr}} - d_{r-1}^{\mathrm{Gr}} - 1 = \left\lceil \frac{d_1}{q^{r-1}} \right\rceil - 1 < \frac{d_1}{q^{r-1}}.$$

Then $d_r < \theta_{r-1}\, d_1/q^{r-1}$, that is, $\sum_{i=0}^{r-1} \lceil d_1/q^i \rceil < \sum_{i=0}^{r-1} \lceil d_1/q^i \rceil$, a contradiction.

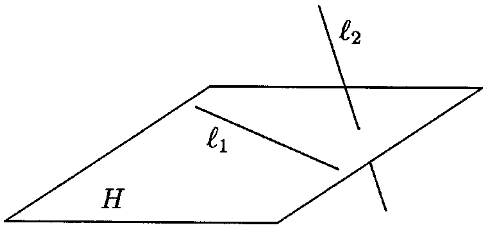

FIGURE 1

The implication (a) $\Rightarrow$ (b) is proved just before Proposition 2 (inequality (24)).

Let us show that (b) $\not\Rightarrow$ (a). Let us present a simple example of a projective system X which does not satisfy the Chain Condition but satisfies the difference Griesmer bound. Note that a number of examples which show this can be found, e.g., in [14], which also give codes not satisfying (b).

Consider a plane H in $\mathbb{P}^3$, a line ℓ_1 in this plane, and a line ℓ_2, which intersects H in a point lying outside ℓ_1. Put $X = (H \backslash \ell_1) \cup \ell_2$ (Fig. 1). Then the minimum-weight plane is H, and the minimum-weight line is ℓ_2, that is, the Chain Condition is not satisfied. Moreover, it is clear that $d_1 = q, d_2 = q - 1$, $d_3 = q^2 + q - 1$, $d_4 = q^2 + q$, that is,

$$\Delta_1 = q, \quad \Delta_2 = q^2 - q - 1 \geq \frac{\Delta_1}{q}, \quad \Delta_3 = q > \frac{\Delta_2}{q}, \quad \Delta_4 = 1 = \frac{\Delta_3}{q}.$$

Finally, let us present a simple example of a projective system which does not satisfy (b). Consider a hyperplane H in $\mathbb{P}^4$, two-dimensional plane H^2 in it, and a line ℓ which intersects H outside H^2 (Fig. 2). Let $X = (H \backslash H^2) \cup \ell$. Then $d_1 = q$, $d_5 = n = q^3 + q$, $d_4 = n - 1 = q^3 + q - 1$, $d_3 = q^3 - 1$, $d_2 = n - q^2 = q^3 - q^2 + q$, whence $\Delta_2 = q^3 - q^2$, $\Delta_3 = q^2 - q - 1$, that is, $\Delta_3 < \Delta_2 / q$ as required. ■

Now, let us return to the problem posed at the beginning of this section. Our main conclusion concerning this problem is contained in the statement below.

PROPOSITION 3. *For a code from hyperbolic quadrics, anticode from elliptic quadrics, Hermitian code for even k, and Hermitian anticode for odd k, all differences of the weight hierarchy* except for one *meet the difference Griesmer bound.*

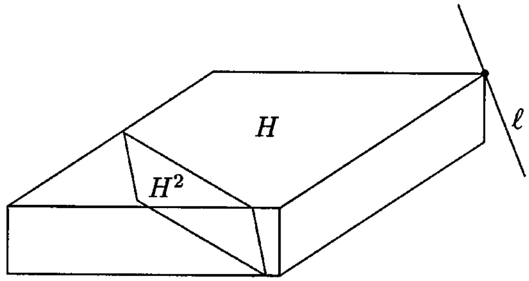

FIGURE 2

Proof. Indeed, [2, formula (6.1)], formula (7) of the present paper, [1, Theorem 5.1], and formula (12) of the present paper imply, respectively, that for a hyperbolic code

$$\Delta_r = \begin{cases} q^{k-r-1}, & r \le k/2, \\ q^{k-r}, & r \ge k/2+1; \end{cases}$$

for an elliptic anticode,

$$\Delta_r = \begin{cases} q^{k-r-1}(q-1), & 1 \le r \le k/2-1, \\ q^{k-r-2}(q^2-1), & k/2 \le r \le k-2, \\ q, & r = k-1, \\ 1, & r = k; \end{cases}$$

for a Hermitian code with even k,

$$\Delta_r = \begin{cases} q^{k-r-1}\sqrt{q}, & r \le k/2, \\ q^{k-r}, & r \ge k/2+1; \end{cases}$$

and for a Hermitian anticode with odd k,

$$\Delta r = \begin{cases} q^{k-r-1}(q-\sqrt{q}), & 1 \le r \le \dfrac{k-1}{2}, \\ q^{k-r-1}(q-1), & \dfrac{k+1}{2} \le r \le k-1, \\ 1, & r = k, \end{cases}$$

whence the desired statement immediately follows. ■

Remark 2. In [1, Remark 1], it is noted that Hermitian codes are in a certain sense the best possible for $r \le k/2$, and on the other hand, for $r \le k/2$ they are the worst since the numbers $n - d_r$ are the largest possible. Our reasoning shows that for $r > k/2$ they are, at the same time, the best possible under given $n - d_{k/2+1}$; in this case, the observation is trivial since the projective system contains planes of dimensions 1 to $k/2 + 1$ (the same takes place for hyperbolic codes as well). However, anticodes from elliptic quadrics and Hermitian varieties for odd k provide examples where this observation is not so trivial.

Moreover, the chains (14)–(23) and the reasoning related yield the corollary below.

COROLLARY 3. *In the weight hierarchy of a code or anticode from a parabolic quadric, Hermitian code for odd k, and Hermitian anticode for even k, starting from $r = 2$, and also a code from an elliptic quadric and an anticode from a hyperbolic quadric, starting from $r = 3$, all differences* except for one *meet the difference Griesmer bound.*

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